Indian Statistical Institute, Bangalore Centre.
Back-paper Exam : Discrete Mathematics I - B2

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Max. points : 40. **Time Limit : 3 hours.**

Give complete proofs. Please cite/quote appropriate results from class. You are also allowed to use results from other problems in the question paper.

Answer any four questions. All questions carry 10 points.

1. Let $0 < a_1 < \ldots < a_{sr+1}$ be $sr+1$ integers. Prove that we can select either $s+1$ of them, no one of which divides any other, or $r+1$ of them with each dividing the following one. Prove that this bound is tight.

2. Let G be a bi-partite graph with partitions $V_1 = \{x_1, \ldots, x_m\}$ and $V_2 = \{y_1, \ldots, y_n\}$. Then G has a subgraph H such that $d_H(x_i) = d_i, d_H(y_j) \leq 1$, $\forall 1 \leq i \leq m$ and $\forall 1 \leq j \leq n$ iff for all $S \subset V_1$, we have that $|N(S)| \geq \sum_{x_i \in S} d_i$. Here $N(S) = \cup_{v \in S} N_v$.

3. Two people play a game on a graph G alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice of the other player. Thus together, the two player are creating a path in the graph. The last player able to move wins the game. Show that the second player has a winning strategy if the graph G has a perfect matching and otherwise the first player has a winning strategy.

4. Let $n \leq 2k$ and $A_1, \ldots, A_m$ be subsets of $[n]$ such that $A_i \cup A_j \neq [n], i, j$. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$.

5. Consider an affine plane of order q i.e., a $2-(q^2, q, 1)$ design. Let a *parallel class* be a set of mutually disjoint lines. Show the following.

   (a) Each parallel class contains q lines ;

(b) there are $q + 1$ such classes ;
(c) any two lines from different classes intersect at a point ;
(d) and lines of each parallel class form a partition of the points $\mathcal{P}$.